# A tour on Riesz means and eigenvalue bounds on Euclidean domains*

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*"The product of mathematics is clarity and understanding, not theorems, by themselves."*

— Bill Thurston

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# 1 Weyl type bounds

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ and consider the Dirichlet Laplacian on $\Omega$:

$$\Delta u + \lambda u = 0, \qquad \text{in } \Omega,$$

with Dirichlet boundary condition on the boundary.

$$u = 0, \qquad \text{on } \partial\Omega.$$

Dirichlet eigenvalues give the frequency of a drum with fixed boundary. The Dirichlet spectrum of $\Omega$ is a sequence of positive real numbers going to infinity and each eigenvalue $\lambda_k$ has finite multiplicity. We order them in an increasing order counting the multiplicity.

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \nearrow \infty$$

Hermann Weyl in 1911 proved the asymptotic behaviour of $\lambda_k$. It states that

$$\lambda_k \sim (2\pi)^2 \omega_d^{-\frac{2}{d}} \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}, \qquad k \nearrow +\infty, \tag{1}$$

where $\omega_d$ is the volume of the $d$-ball,

$$\omega_d = \pi^{d/2}/\Gamma(1 + d/2).$$

For any $z > 0$, let $N(z)$ be the number of eigenvalues, counted with multiplicity, that are strictly less than $z$:

$$N(z) = \sup\{k \in \mathbb{N} : \lambda_k < z\} = \#\{k \in \mathbb{N} : \lambda_k < z\}$$

Notice that

$$\lambda_k = \inf\{z > 0 : N(z) \geqslant k\}.$$

The function $N(z)$ is called the counting function. The Weyl law (1) can be re-written as

$$N(z) \sim \frac{\omega_d}{(2\pi)^d}|\Omega| z^{d/2} = L^{cl}_{0,d}\,|\Omega| z^{d/2}, \qquad z \nearrow \infty \tag{2}$$

where $L^{cl}_{0,d} := \omega_d/(2\pi)^d = \frac{1}{(4\pi)^{d/2}\Gamma(1+d/2)}$ is called the *classical constant*.

Note that the asymptotics (1) and (2) also hold for the eigenvalues of the Neumann Laplacian on a bounded domain $\Omega$ of $\mathbb{R}^d$ with Lipschitz boundary. We recall that the Neumann eigenvalue problem is

$$\Delta u + \lambda u = 0, \qquad \text{in } \Omega,$$

with Neumann boundary condition on the boundary:

$$\partial_\nu u = 0, \qquad \text{on } \partial\Omega,$$

where $\partial_\nu u$ is the normal derivative of $u$ along the boundary. We denote its eigenvalues by $\mu_k$.

$$0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots \nearrow +\infty.$$

## 1.1 Pólya Conejcture

G. Pólya conjectured that the Weyl formula (1) not only describes the asymptotic behaviour of $\lambda_k$ but also gives a lower bound for $\lambda_k$ for every $k \geqslant 1$:

$$\lambda_k \geqslant (2\pi)^2 \omega_d^{-\frac{2}{d}} \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}.$$

For the Neumann eigenvalues he conjectured that the Weyl asymptotic gives an upper bound for $\mu_k$ for every $k \geqslant 1$:

$$\mu_{k+1} \leqslant (2\pi)^2 \omega_d^{-\frac{2}{d}} \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}.$$

In 1961, Pólya in his paper [26] dedicated to H. Weyl proved his conjecture for the Dirichlet eigenvalues on a tiling domain and for the Neumann eigenvalues on a regularly tiling domain in $\mathbb{R}^2$. A domain $\Omega \subset \mathbb{R}^2$ is called to be a *tiling domain* or a *plane-covering domain* if infinitely many copies of $\Omega$ can cover the whole plane without gaps and without overlapping. Any triangle, any quadrilateral, and any hexagon with a center are examples of tiling domains. But a disk in not a tiling domain. A domain $\Omega$ is regularly tiling if we can translationally cover the plane by a domain constructed by a finite disjoint copies of $\Omega$. The same idea of proof shows that the Pólya conjecture holds for tiling domains in $\mathbb{R}^d$. Laptev in [16] showed that the Pólya conjecture for the Dirichlet eigenvalues on $\Omega$ holds if $\Omega$ is of the form $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d_1 \geqslant 2$, where $\Omega_1$ satisfies the Pólya conjecture and $\Omega_2$ is orbitary (for a similar result for the Neumann eigenvalues see [10]). However, the conjecture remains open even for a disk!

### 1.2 Pólya's proof for a tiling domain

The proof of Pólya for a tiling domain $\Omega$ is simple and nice. It essentially uses the fact that $\Omega$ is tiling. Beside this fact, only the Weyl asymptotic for a square and the monotonicity of eigenvalues are used in the proof. Let $D$ be a square of side 1. By the Weyl asymptotic we know that $\lim_{k\to+\infty} \lambda_k(D)k^{-1} = 4\pi$.

**Exercise 1.1.** *Prove the eigenvalue asymptotic of a square in $\mathbb{R}^2$ without knowing the Weyl law.*

Let $\delta > 0$ and $\Omega_\delta$ be the scaled version of $\Omega$ by the factor $\delta$. We use the following simple facts

$$\lambda_k(\Omega_\delta) = \delta^{-2}\lambda_k(\Omega); \qquad |\Omega_\delta| = \delta^2|\Omega|.$$

Let cover the plane by $\Omega_\delta$ and let $n$ be the number of $\Omega_\delta$ contained completely in $D$. We can consider $\delta$ as a function of $n$. Note that when $n \to \infty$, then $\delta$ tends to zero.

$$\lim_{n\to\infty} n|\Omega_\delta| = \lim_{n\to\infty} n\delta^2|\Omega| = |D| = 1.$$

By monotonicity of eigenvalues, $\lambda_{nk}(D) \leqslant \lambda_k(\Omega_\delta) = \delta^{-2}\lambda_k(\Omega)$. We conclude

$$\lambda_k(\Omega) \geqslant \lim_{n\to\infty} \delta^2 nk(nk)^{-1}\lambda_{nk}(D) = 4\pi k.$$

**Exercise 1.2.** *Read the proof of Pólya in the Neumann case in [26, Section 5] and see where the assumption of having a regulated tiling domain instead of a tiling domain is needed.*

**Exercise 1.3.** *Let $\{S_i\}_{i\in\mathbb{N}}$ be a collection of identical mutually disjoint disks in $\mathbb{R}^2$. Define the packing density of this collection as*

$$\eta = \lim_{t\to\infty} \frac{\sum_{i=1}^{\infty} |S_i \cap D_t|}{|D_t|}$$

*where $D_t$ is a square of side $t$. Let $\eta_0$ be the best packing density among all collections of disks in $\mathbb{R}^2$. Can you mimic the same idea of Pólya's proof and get some lower bound for $\lambda_k$ on a disk?*

### 1.3 Berezin–Li–Yau inequality

Let us consider the sum of first $k$ eigenvalues $\sum_{j=1}^k \lambda_j$. We can easily obtain that the Weyl asymptotic of $\sum_{j=1}^k \lambda_j$ from (1):

$$\sum_{j=1}^{k} \lambda_j \sim (2\pi)^2 \omega_d^{-d/2} \frac{d}{d+2} \frac{k^{1+2/d}}{|\Omega|^{2/d}}, \qquad k \nearrow \infty\,. \tag{3}$$

While the Pólya conjecture remains widely open, in 1983, Li and Yau proved that if instead of considering an individual eigenvalue $\lambda_k$ one considers $\sum_{j=1}^k \lambda_j$ then the Weyl asymptotic (3) gives a lower bound [22]:

$$\sum_{j=1}^{k} \lambda_j \geqslant (2\pi)^2 \omega_d^{-d/2} \frac{d}{d+2} \frac{k^{1+2/d}}{|\Omega|^{2/d}}, \tag{4}$$

One can immediately get that

$$\lambda_k \geqslant (2\pi)^2 \omega_d^{-d/2} \frac{d}{d+2} \frac{k^{2/d}}{|\Omega|^{2/d}}, \tag{5}$$

and in terms of the counting function $N(z)$:

$$N(z) \leqslant \left(\frac{d+2}{d}\right)^{d/2} L_{0,d}^{cl}\, |\Omega| z^{d/2}. \tag{6}$$

We see that (5) differes from the bound in the Pólya conjecture by the factor of $\frac{d}{d+2}$. The inequality of the form

$$\lambda_k \geqslant C_d (2\pi)^2 \omega_d^{-d/2} \frac{k^{2/d}}{|\Omega|^{2/d}},$$

with $0 < C_d < \frac{d}{d+1}$ was proved previously, see for example [23, 25]. So far (5) is the best known lower bound.
It turned out that the Li–Yau inequality can be obtained as a corollary of an inequality proved by Berezin in 1972 [1]:

$$R(z) := \sum_k (z - \lambda_k)_+ \leqslant L_{1,d}^{cl} z^{1+d/2} |\Omega|, \tag{7}$$

where $L_{1,d}^{cl} = \frac{2}{2+d} L_{0,d}^{cl}$. In the next section we give the definition of $L_{\rho,d}^{cl}$ for any $\rho \geqslant 0$. Laptev and Weidl in [18] showed that by applying the Legendre transform to (7) we can get the Li–Yau inequality. They refer to these inequalities as the Berezin–Li–Yau inequality. The function $R(z)$ is called the Riesz mean. It is one of the key objects of our focus in this note. In the next section we discuss properties of the Riesz means.

Let us see how one can get the Li-Yau inequality as a consequence of the Berezin inequality.

**Definition 1.1** (Legendre Transform)**.** *Let $\varphi : I \to \mathbb{R}$ be a convex function defined on interval $I \subset \mathbb{R}$. The Legendre transform of $\varphi$ is the function $\varphi^* : J \to \mathbb{R}$ defined by*

$$\varphi^*(a) = \sup_{x \in I} (ax - \varphi(x)), \quad a \in J$$

*where $J = \{a \in \mathbb{R} : \sup_{x \in I}(ax - \varphi(x)) < \infty\}$. The Legendre transform is always well-defined when $\varphi(x)$ is convex.*

Let us calculate the Legendre transform of $R(z)$ and the right-hand side of Inequality (7),

$$f(z) := \frac{2}{2+d} L_{0,d}^{cl} z^{1+d/2} |\Omega|).$$

We get

$$R^*(a) = \sup_{z \geqslant 0} \{az - \sum_k (z - \lambda_k)_+\} = (a - [a])\lambda_{[a]+1} + \sum_{k=1}^{[a]} \lambda_k\,, \tag{8}$$

and

$$f^*(a) = \frac{d}{2+d} \left(L_{0,d}^{cl} |\Omega|\right)^{-\frac{2}{d}} a^{1+\frac{2}{d}}.$$

**Exercise 1.4.** *Prove identity* (8)*.*

Since $R(z) \leqslant f(z)$, $z \geqslant 0$, we have $R^*(a) \geqslant f^*(a)$, $a \geqslant 0$. If we take $a = k \in \mathbb{N}$, then we immediately recover the Li–Yau inequality (4).

**Theorem 1** (Berezin–Li–Yau inequality)**.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$. Then*

$$R(z) \leqslant L_{1,d}^{cl} z^{1+d/2} |\Omega|. \tag{9}$$

We postpone the proof of the Berezin–Li–Yau inequality to Section 2.
The Berezin –Li–Yau inequality (9) can be written in the general form of

$$R_\rho(z) \leqslant L^{cl}_{\rho,d} z^{1+d/2} |\Omega|. \tag{10}$$

We define the Riesz mean $R_\rho(z)$ of order $\rho$ and the classical constant $L^{cl}_{\rho,d}$ below. These bounds are Weyl type. In general, when the asymptotics of the quantity under the consideration appear in the upper/lower bounds, we call such type of bound to be of the *Weyl type*.
Laptev in [16] gave a simple and nice proof of Weyl type upper bounds for a more general class of operators (e.g. $(-\Delta)^\alpha$, $\alpha > 0$) using a general (sharp) trace inequalities for convex functions of operators. We discuss the proof of Laptev in Section 2.
As an application of Berezin–Li–Yau inequality (9), Laptev in [16] proved the following interesting result.

**Theorem 2.** *Let $\Omega \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be of the form $\Omega = \Omega_1 \times \Omega_2$ where $d_1 \geqslant 2$, $d_2 \geqslant 1$. Suppose that the Dirichlet Laplacian on $\Omega_1 \subset \mathbb{R}^{d_1}$ satisfies the Pólya conjecture and $\Omega_2 \subset \mathbb{R}^{d_2}$ is an arbitrary domain of finite measure. Then*

$$\lambda_k \geqslant (2\pi)^2 \omega_d^{-\frac{2}{d}} \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}, \qquad d := d_1 + d_2,$$

*or equivalently*

$$N(z, -\Delta_D) \leqslant L^{cl}_{0,d} |\Omega| z^{\frac{d}{2}}.$$

*Proof.* We only present the proof for $d_1 = 2$. The proof for $d_1 > 2$ is similar and you can find the details in [16, Theorem 2.8].
Let $(\lambda^{(1)}_k)_{k\geqslant 1}$ and $(\lambda^{(2)}_k)_{k\geqslant 1}$ be the sequence of eigenvalues of the Dirichlet Laplacian on $\Omega_1$ and $\Omega_2$ respectively. Then the eigenvalues $\lambda_{mn}$ of the Dirichlet Laplacian $-\Delta$ on $\Omega$ is of the form.

$$\lambda_{mn} = \lambda^{(1)}_m + \lambda^{(2)}_n, \qquad m, n \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} N(z, -\Delta_D) &= \#\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \lambda^{(1)}_m + \lambda^{(2)}_n < z\right\} \\ &= \#\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \lambda^{(1)}_m < (z - \lambda^{(2)}_n)_+\right\} \\ &\leqslant L^{cl}_{0,d_1} |\Omega_1| \sum_n \left(z - \lambda^{(2)}_n\right)^{\frac{d_1}{2}}_+ . \end{aligned}$$

The last inequality is the consequence the fact that the Pólya conjecture holds on $\Omega_1$. Note that $d_1 = 2$. Now we use inequality (9):

$$N(z, -\Delta_D) \leqslant L^{cl}_{0,2} L^{cl}_{1,d_2} |\Omega_1||\Omega_2| z^{1+\frac{d_2}{2}} = L^{cl}_{0,d} |\Omega| z^{\frac{d}{2}}.$$

Notice that we only prove the theorem for $d_1 = 2$. □

### 1.4 Riesz Means

The Riesz mean of order $\rho > 0$ is defined, for any $z \geqslant 0$, by

$$R_\rho(z) = \sum_k (z - \lambda_k)^\rho_+ ,$$

where $(z-\lambda)_+ := \max(0, z-\lambda)$. For simplicity we denote $R_1(z)$ by $R(z)$. When $\rho \to 0+$, the Riesz mean reduces to the counting function

$$N(z) = \sum_{\lambda_k < z} 1 = \sup\{k : \ \lambda_k < z\}.$$

Even though, the Riesz mean is only defined for $\rho > 0$, by convention, we sometimes write $N(\lambda)$ as

$$N(z) = R_0(z) = \sum_{\lambda_k < z} (z - \lambda_k)_+^0 .$$

Note that $R_0(z)$ is a step function and discontinuous. But $R_\rho(z)$ for every $\rho > 0$ is a continuous function. Moreover, for $\rho \geqslant 1$, $R_\rho(z)$ is differentiable with continuous derivative.

**Some properties of the Riesz mean**

The book[1] by Chandrasekharan and Minakshisundaram [2] contains fundamental properties of the Riesz mean where they refer to it as the typical means. Here we list some main properties of $R_\rho(z)$. There are relations between Riesz means of different orders. Let $\rho, \delta > 0$. Then

$$R_{\rho+\delta}(z) = \frac{\Gamma(\rho+\delta+1)}{\Gamma(\rho+1)\,\Gamma(\delta)} \int_0^\infty (z-t)_+^{\delta-1} R_\rho(t) dt. \tag{11}$$

This is called the *Riesz iteration*.

**Exercise 1.5.** *Recall the definition of Euler integrals* $\Gamma$ *and* $\mathcal{B}$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \qquad \mathcal{B}(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt.$$

*Prove the Riesz iteration formula using the following identity.*

$$\int_0^\infty (1-t)_+^{p-1}\, t^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \mathcal{B}(p,q).$$

**Exercise 1.6.** *Show that*

$$R(z) = \int_0^z N(z) dz.$$

*We conclude that the Riesz iteration* (11) *is true for* $\rho \geqslant 1$ *and* $\delta \geqslant 0$.

**Weyl asymptotic of $R_\rho(z)$**

One can immediately get the asymptotics of $R_\rho(z)$ by using the Riesz iteration of (1):

$$R_\rho(z) \sim L^{cl}_{\rho,d}\, |\Omega|\, z^{\rho+d/2} \quad \text{as } z \to \infty, \tag{12}$$

where the classical constant is given by

$$L^{cl}_{\rho,d} = \frac{\Gamma(1+\rho)}{(4\pi)^{d/2}\,\Gamma(1+\rho+d/2)}. \tag{13}$$

**Exercise 1.7.** *Having* (9)*, show that*

$$R_\rho(z) \leqslant L^{cl}_{\rho,d}\, |\Omega|\, z^{\rho+d/2}.$$

## 1.5 Kröger inequality

In 1992, Kröger [15] proved a counterpart of the Berezin–Li–Yau inequality for the Neumann eigenvalues. More precisely, he showed that

$$\sum_{j=1}^k \mu_j \leqslant (2\pi)^2 \omega_d^{-d/2} \frac{d}{d+2} \frac{k^{1+2/d}}{|\Omega|^{2/d}}, \tag{14}$$

Laptev in [16] showed that the Kröger inequality can be derived from an inequality for the Riesz means:

[1]This book is available on the online library archive.org for free. You can find it in the following address https://archive.org/details/typicalmeans032098mbp

**Theorem 3.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary. Then*

$$R_\rho(z,-\Delta_N)=\sum_k (z-\mu_k)_+^\rho \geqslant L^{cl}_{\rho,d} z^{\rho+d/2}|\Omega|. \tag{15}$$

Recall the asymptotics of $\sum_{j=1}^k \mu_j$ and $R(z,-\Delta_N)$ in (3) and (12) and observe that these bounds are Weyl type.

**Exercise 1.8.** *Using the Legendre transform show that* (14) *is a consequence of* (15) *for* $\rho=1$.

**Corollary 1.1.** *Under the assumption of Theorem* (3)*, we have*

$$\mu_{k+1}\leqslant (2\pi)^2\omega_d^{-\frac{2}{d}}\left(\frac{d+2}{2}\right)^{\frac{2}{d}}\left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}},\qquad k\geqslant 1.$$

*Proof.* See Exercise (2.3).

□

# 2 Proof of eigenvalue bounds

In the section, we prove the Berezin–Li–Yau inequality (9) and the Kröger inequality (15).

## 2.1 Proof of the Berezin–Li–Yau inequality

We follow the simple proof given in [16] (see also [17]). Let $\{\phi_k\}$ be an orthonormal sequence of eigenfunction of the Dirichlet Laplacian $-\Delta$. We can write

$$\begin{aligned}
\sum_k (z-\lambda_k)_+ &= \sum_k \left(\langle (z-\lambda_k)\phi_k,\phi_k\rangle\right)_+\\
&= \sum_k \left(\langle (z+\Delta)\phi_k,\phi_k\rangle\right)_+\\
&= (2\pi)^{-d}\sum_k\left(\int_{\mathbb{R}^d}(z-|\xi|^2)|\hat{\phi}_k(\xi)|d\xi\right)_+\\
&\leqslant (2\pi)^{-d}\sum_k\int_{\mathbb{R}^d}(z-|\xi|^2)_+|\hat{\phi}_k(\xi)|^2d\xi
\end{aligned}$$

where $\langle f,g\rangle=\int_\Omega f\bar{g}$. Recall that $|\hat{\phi}_k(\xi)|^2=|\int_\Omega e^{ix\xi}\phi_k(x)dx|^2$. Since $\{\phi_k\}$ is an orthonormal basis of $L^2(\Omega)$, $|\hat{\phi}_k(\xi)|^2$ can be considered as a Fourier coefficient of $e^{ix\xi}$. Thus

$$\sum_k |\hat{\phi}_k(\xi)|^2=\int_\Omega |e^{ix\xi}|^2dx=|\Omega|.$$

Therefore,

$$\begin{aligned}
\sum_k (z-\lambda_k)_+ &\leqslant (2\pi)^{-d}\int_{\mathbb{R}^d}(z-|\xi|^2)_+\sum_k|\hat{\phi}_k(\xi)|^2d\xi\\
&= (2\pi)^{-d}|\Omega|\int_{\mathbb{R}^d}(z-|\xi|^2)_+d\xi\\
&= (2\pi)^{-d}|\Omega|z^{\frac{d}{2}+1}d\,\omega_d\int_0^1(1-r^2)r^{d-1}dr=L^{cl}_{1,d}|\Omega|z^{1+d/2}
\end{aligned}$$

**Exercise 2.1.** *This exercise gives you a simple proof on how to get inequality* (6) *for the counting function $N(z)$ from inequality* (9)*.*

*(i) Show that for any $0 < \alpha < \beta$ we have.*

$$N(\beta - \alpha, -\Delta_D) \leqslant \frac{1}{\alpha} \sum_k (\beta - \lambda_k)$$

*(Hint: verify $1 \leqslant \frac{1}{\alpha}(\beta - x)_+$ for any $x \in (0, \beta - \alpha)$.)*

*(ii) Show that*

$$N(z) \leqslant \left(\frac{d+2}{d}\right)^{d/2} L_{0,d}^{cl} \, |\Omega| z^{d/2}.$$

*(Hint: take $\alpha = \frac{2}{d} z$ and $\beta = (1 + \frac{2}{d})z$ in (i).)*

### 2.2 Averaged variational principle

The proof of the Kröger inequality and inequality (15) can be formulated in an abstract version. This abstract version was introduced in a recent paper [4, Theorem 2.1] by El Soufi, Harrell, Ilias, and Stubbe. We state this theorem for the Neumann Laplacian. Note that the same statement holds for a large class of eigenvalue problems.

**Theorem 4** (Averaged variational principle)**.** *Let $\{\phi_k\}$ be an orthonormal sequence of eigenfunctions of $-\Delta$. Let $f_\xi \in H^1(\Omega)$ be a family of functions whose index $\xi$ varies over a measure space $(\mathfrak{M}, \Sigma, \mu)$, with $\sigma$-algebra $\Sigma$. Let $\mathfrak{M}_0$ be a measurable subset of $\mathfrak{M}$. Then for any $z \in \mathbb{R}_+$ we have*

$$\sum_j (z - \mu_j)_+ \int_{\mathfrak{M}} |\langle \varphi_j, f_\xi \rangle|^2 d\mu \geqslant \int_{\mathfrak{M}_0} \left( z \int_\Omega |f_\xi(x)|^2 dx - \int_\Omega |\nabla f_\xi(x)|^2 dx \right) d\mu. \tag{16}$$

*Proof.* The proof of this theorem is a consequence of the Rayleigh theorem for the eigenvalues. See for example [3]. It states that $\mu_{k+1}$ can be characterised by

$$\mu_{k+1} = \inf_{0 \neq f \in V_k} R(f), \tag{17}$$

where $V_k = \big\{ g \in H^1(\Omega) : \langle g, \phi_j \rangle = 0, j = 1, \dots, k \big\}$, and

$$R(f) := \frac{\int_\Omega |\nabla f|^2 dx}{\int_\Omega |f|^2 ds}.$$

Given $f_\xi \in H^1(\Omega)$, note that $f_\xi - \sum_{j=1}^{k-1} \langle \phi_j, f_\xi \rangle \phi_j \in V_k$. Take $z \in (\mu_k, \mu_{k+1}]$. Thus, by (17) we have

$$z \leqslant R\left( f_\xi - \sum_{j=1}^{k} \langle \phi_j, f_\xi \rangle \phi_j \right) = \frac{\int_\Omega |\nabla f_\xi|^2 \, dx - \sum_{j=1}^k \mu_j |\langle \phi_j, f_\xi \rangle|^2}{\int_\Omega |f_\xi|^2 \, dx - \sum_{j=1}^k |\langle \phi_j, f \rangle|^2}.$$

Therefore,

$$\sum_j (z - \mu_j)_+ \, |\langle \phi_j, f_\xi \rangle|^2 \geqslant z \int_\Omega |f_\xi|^2 ds - \int_\Omega |\nabla f_\xi|^2 dx. \tag{18}$$

We conclude

$$\begin{aligned} \sum_j (z - \mu_j)_+ \int_{\mathfrak{M}} |\langle \phi_j, f_\xi \rangle|^2 d\mu &\geqslant \sum_j (z - \mu_j)_+ \int_{\mathfrak{M}_0} |\langle \varphi_j, f_\xi \rangle|^2 d\mu \\ &\geqslant \int_{\mathfrak{M}_0} \left( z \int_\Omega |f_\xi(x)|^2 dx - \int_\Omega |\nabla f_\xi(x)|^2 dx \right) d\mu. \end{aligned}$$

□

### 2.3 Proof of the Kröger inequality

We now show that the inequality

$$R(z,-\Delta_N)=\sum_k (z-\mu_k)_+ \geqslant L^{cl}_{1,d} z^{1+d/2}|\Omega|$$

is an application of the averaged variational principle. Indeed, it is enough to take $f_\xi(x)=e^{ix\xi}$ where $\xi\in\mathbb{R}^d$. Then by Parceval's identity

$$(2\pi)^{-d}\int_{\mathbb{R}^d}|\langle\phi_j,f_\xi\rangle|^2 d\xi=\int_{\mathbb{R}^d}|\hat{\phi}_j(\xi)|^2 d\xi=1.$$

Now take $\mathfrak{M}_0=\{\xi\in\mathbb{R}^d : |\xi|\leqslant\sqrt{z}\}$. We get,

$$\begin{aligned}\sum_j (z-\mu_j)_+ &\geqslant (2\pi)^{-d} z\int_{|\xi|\leqslant\sqrt{z}}\left(\int_\Omega |e^{ix\xi}|^2dx-\int_\Omega|\nabla(e^{ix\xi})|^2dx\right)d\xi\\ &= (2\pi)^{-d}|\Omega|\int_{|\xi|\leqslant\sqrt{z}}\left(z-|\xi|^2\right)d\xi\\ &= (2\pi)^{-d}|\Omega| z^{\frac{d}{2}+1} d\,\omega_d\int_0^1(1-r^2)r^{d-1}dr=L^{cl}_{1,d}|\Omega|z^{1+d/2}.\end{aligned}$$

**Exercise 2.2.** *Having the lower bound above on $R(z,-\Delta_N)$, obtain a lower bound on $R_\rho(z,-\Delta_N)$ by the means of the Riesz iteration.*

**Exercise 2.3.** *(i) Show that the counting function $N(z,-\Delta_N)$ for the Neumann eigenvalues satisfies*

$$N(z,-\Delta_N)\geqslant L^{cl}_{1,d}|\Omega| z^{\frac{d}{2}}.$$

*Hint: use the trivial inequality $\frac{1}{z}(z-x)\leqslant 1$ for every $x\in[0,z)$.*

*(ii) Show that*

$$\mu_{k+1}\leqslant (2\pi)^2\omega_d^{-\frac{2}{d}}\left(\frac{d+2}{2}\right)^{\frac{2}{d}}\left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}.$$

*Recall that $\mu_{k+1}=\inf\{z\geqslant 0 : N(z)>k\}$.*

## 3 Equivalence of Kac's and Berezin–Li–Yau's inequalities

Harell and Hermi [8] showed several equivalence between the Riesz means inequalities and other bounds for the eigenvalues of the Dirichlet Laplacian. These inequalities have been proved by independent methods. Here we study one set of these equivalent inequalities.

We need to first define an important function of eigenvalues, the trace of the heat kernel:

$$Z(t):=\sum_{k=1}^\infty e^{-\lambda_k t}. \tag{19}$$

The homogenous heat equation is

$$\begin{cases}(-\Delta+\partial_t)u=0 & \text{in } \Omega\times(0,\infty)\\ u(\cdot,0)=f & \text{on } \Omega\times\{0\}\end{cases}$$

where $u(x,t)$ represents the temperature at point $x\in\Omega$ at time $t$, and $f(x)$ is the initial temperature. Let $p(x,y,t)$ be the fundamental solution of the heat equation, i.e. $(-\Delta+\partial_t)p(\cdot,y,t)=0$ and $\lim_{t\to 0}p(\cdot,y,t)=\delta_y$. Then the function $u(x,,t)=\int_\Omega p(x,y,t)f(y)dy$ solves the heat equation. Let $\{\phi_k\}$ be an orthonormal sequence of the Laplace eigenfunctions. We have

$$p(x,y,t)=\sum_{j=1}^\infty e^{-\lambda_j t}\phi_j(x)\phi_j(y). \tag{20}$$

**Exercise 3.1.** *Read the proof of identity* (20) *in Chavel's book [3] for example.*

In this section we want to study the relation between the bounds for the trace of the heat kernel

$$\int_\Omega p(x,x,t)dx = \sum_{k=1}^{\infty} e^{-\lambda_k t} =: Z(t)$$

and the Berezin–Li–Yau inequality.

The asymptotic formula of $Z(t)$ is given by

$$Z(t) \sim \frac{|\Omega|}{(4\pi t)^{d/2}}, \qquad t \to 0. \tag{21}$$

It is proved by Kac in 1951 [12]. Later in his well-known paper [13] "Can one hear the shape of a drum?", he proved the following inequality for $Z(t)$:

$$Z(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \leqslant \frac{|\Omega|}{(4\pi t)^{d/2}}. \tag{22}$$

The main goal is to prove

**Theorem 5** ([8])**.** *The Kac inequality is equivalent to the Berezin–Li–Yau inequality for $\rho \geqslant 2$, i.e.*

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \leqslant \frac{|\Omega|}{(4\pi t)^{d/2}} \quad \Leftrightarrow \quad R_\rho(z) \leqslant L^{cl}_{\rho,d} z^{1+d/2}|\Omega|, \text{ for any } \rho \geqslant 2.$$

*Proof.* "$\Leftarrow$" The proof is an application of the Laplace transform. Recall that

$$\mathcal{L}(f)(t) = \int_0^\infty f(z)e^{-tz}\,dz.$$

**Exercise 3.2.** *Let $\psi_\lambda(z) := (z-\lambda)^\rho_+$. Show that*

$$\mathcal{L}(\psi_\lambda)(t) = \frac{\Gamma(\rho+1)\ e^{-\lambda\ t}}{t^{\rho+1}}.$$

Thus we have

$$\mathcal{L}(R_\rho)(t) = \mathcal{L}\left(\sum_k \psi_{\lambda_k}\right)(t) = \frac{\Gamma(\rho+1)\sum_k e^{-\lambda_k\ t}}{t^{\rho+1}}. \tag{23}$$

Taking the Laplace transform of the right-hand side of the Berezin–Li–Yau inequality leads to

$$\frac{\Gamma(\rho+1)}{t^{\rho+1}} Z(t) \leqslant L^{cl}_{\rho,d}\,|\Omega| \frac{\Gamma(\rho+1+\frac{d}{2})}{t^{\rho+1+\frac{d}{2}}},$$

After rearranging we get

$$Z(t) \leqslant \frac{|\Omega|}{t^{\frac{d}{2}}} \frac{L^{cl}_{\rho,d}\,\Gamma(\rho+1+\frac{d}{2})}{\Gamma(\rho+1)} = \frac{|\Omega|}{(4\pi t)^{d/2}}.$$

To get the last identity we only use the definition of the classical constant $L^{cl}_{\rho,d}$. It is clear that we only need to now the Berezin–Li–Yau inequality for one value of $\rho$.

"$\Rightarrow$" We now show that the Kac inequality gives the Berezin–Li–Yau inequality for $\rho \geqslant 2$. A crucial fact that we use here is the following result of Harrell and Hermi in [7]: The function

$$z \mapsto \frac{R_\rho(z)}{z^{\rho+\frac{d}{2}}} \tag{24}$$

is a nondecreasing function of $z$, for $\rho \geqslant 2$.
Take $z_0 \geqslant 0$. By the monotonicity result (24) we have

$$R_\rho(z+z_0) \geqslant R_\rho(z_0)\left(\frac{z+z_0}{z_0}\right)^{\rho+d/2}. \tag{25}$$

We now apply the Laplace transform on both side of the inequality.

**Exercise 3.3.** *Check that the Laplace transform of a shifted function $f(z+z_0)$ is given by the formula*

$$\mathcal{L}\left(f(\cdot+z_0)\right)(t) = e^{z_0\, t}\left(\mathcal{L}(f)(t) - \int_0^{z_0} e^{-tz} f(z)dz\right).$$

Therefore,

$$\mathcal{L}\left(\psi_{\lambda_k}(\cdot+z_0)\right)(t) = e^{(z_0-\lambda_k)_+ t}\left(\frac{\Gamma(\rho+1)}{t^{\rho+1}} - \int_0^{(z_0-\lambda_k)_+ t} e^{-tz} z^{\rho} dz\right), \tag{26}$$

and

$$\mathcal{L}\left((\cdot+z_0)^{\rho+d/2}\right)(t) = e^{z_0\, t}\left(\frac{\Gamma(\rho+1+d/2)}{t^{\rho+1+d/2}} - \int_0^{z_0\, t} e^{-tz} z^{\rho+d/2} dz\right). \tag{27}$$

whereas in the second term of (26) and (27) we have the lower incomplete Gamma function

$$\gamma(a,x) = \int_0^x e^{-z} z^{a-1} dz.$$

So the Laplace transform of inequality (25) is

$$\sum_k e^{(z_0-\lambda_k)_+ t}\left(\frac{\Gamma(\rho+1)}{t^{\rho+1}} - \gamma\left(\rho+1,(z_0-\lambda_k)_+ t\right)\right) \geqslant \frac{R_\rho(z_0)}{z_0^{\rho+d/2}} e^{z_0\, t}\left(\frac{\Gamma(\rho+1+d/2)}{t^{\rho+1+d/2}} - \gamma(\rho+1+d/2, z_0\, t)\right).$$

Observe that

$$\sum_k e^{(z_0-\lambda_k)_+ t} \leqslant e^{z_0\, t} \sum_{k=1}^{\infty} e^{-\lambda_k t} = e^{z_0\, t}\, Z(t). \tag{28}$$

Therefore,

$$\frac{\Gamma(\rho+1)}{\Gamma(\rho+1+d/2)}\, t^{d/2} Z(t) \geqslant \frac{R_\rho(z_0)}{z_0^{\rho+d/2}} + \mathcal{R}(t), \tag{29}$$

where we have an explicit formula for the remainder term $\mathcal{R}(t)$. One can check that $\lim_{t\to 0}\mathcal{R}(t) = 0$ Taking the limit as $t \to 0$ and using the asymptotic (21)we get

$$L^{cl}_{\rho,d}|\Omega| = \frac{\Gamma(\rho+1)}{(4\pi)^{d/2}\,\Gamma(\rho+1+d/2)}\,|\Omega| \geqslant \frac{R_\rho(z_0)}{z_0^{\rho+d/2}}, \tag{30}$$

which shows that the Berezin-Li-Yau follows for $\rho \geqslant 2$. □

## 3.1 More discussion

Here we briefly mention another spectral function of interest, the spectral zeta function $\zeta_{spec}$. It is defined by

$$\zeta_{spec}(\rho) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\rho}}.$$

From the inequality

$$\lambda_k \geqslant (2\pi)^2 \omega_d^{-d/2} \frac{d}{d+2} \frac{k^{2/d}}{|\Omega|^{2/d}},$$

we can get an immediate upper bound on $\zeta_{spec}(\rho)$

$$\zeta_{spec}(\rho) \leqslant \left(\frac{d+2}{d}\right)^{\rho} \frac{\zeta(2\rho/d)}{(4\pi^2)^{\rho}}\ \left(\omega_d\,|\Omega|\right)^{2\rho/d},$$

where $\zeta(\rho) = \sum_{k=1}^{\infty} \frac{1}{k^\rho}$ is the zeta function.

An integral transform which relates the bounds on the spectral zeta function $\zeta_{spec}(\rho)$ to the bounds on $Z(t)$ is the Mellin transform. The Mellin transform of a function $f$ is given by

$$\mathcal{M}f(s) = \int_0^{\infty} x^{s-1} f(x)\, dx.$$

It is easy to check that

$$\zeta_{spec}(\rho) = \frac{1}{\Gamma(\rho)} \int_0^\infty t^{\rho-1} Z(t) dt.$$

Indeed by the definition of $\Gamma$ function we have

$$\frac{1}{\lambda^\rho} = \frac{1}{\Gamma(\rho)} \int_0^\infty e^{-\lambda t} t^{\rho-1} dt.$$

It is conjectured that the Kac inequality can be improved by a factor $e^{-\frac{t}{|\Omega|^{2/d}}}$:

**Conjecture 1** ( Improvement of the Kac inequality)**.**

$$Z(t) = \sum_{j=1}^\infty e^{-\lambda_j t} \leqslant \frac{|\Omega|}{(4\pi t)^{d/2}} e^{-\frac{t}{|\Omega|^{2/d}}}. \tag{31}$$

By applying the Mellin transform we get the following conjecture for the upper bound of $\zeta_{spec}(\rho)$.

**Conjecture 2.** *For $\rho > d/2$,*

$$\zeta_{spec}(\rho) \leqslant \frac{\Gamma(\rho - d/2)}{\Gamma(\rho)} \frac{|\Omega|^{2\rho/d}}{(4\pi)^{d/2}}. \tag{32}$$

Some improvement of the Kac inequality can be obtained by applying the Laplace transform on the refined Berezin–Li–Yau inequality, see Exercise 4.2.This also leads to an improvement of bounds on $\zeta_{spec}(\rho)$.

In previous section we have seen how integral transforms relate different spectral functions. For more discussion we refer to [7, 8].

# 4 Two term bounds on Riesz means and open problems

After celebrated work of Weyl on asymptotics of the Laplace eigenvalues, there are numerous work on obtaining the second term in the asymptotic. The main result in this direction goes back to Ivrii [11]. He proved that under some appropriate conditions on $\Omega$ and its boundary $\partial\Omega$ the following holds

$$N(z, -\Delta_D) = L^{cl}_{0,d} |\Omega| z^{d/2} - \frac{1}{4} L^{cl}_{0,d-1} |\partial\Omega| z^{(d-1)/2} + o(z^{(d-1)/2}), \qquad z \nearrow \infty, \tag{33}$$

and more generally for $R_\rho(z)$, $\rho \geqslant 0$, we have

$$R_\rho(z, -\Delta_D) = L^{cl}_{\rho,d} |\Omega| z^{\rho+d/2} - \frac{1}{4} L^{cl}_{\rho,d-1} |\partial\Omega| z^{\rho+(d-1)/2} + o(z^{\rho+(d-1)/2}), \qquad z \nearrow \infty. \tag{34}$$

And for the Neumann Laplacian

$$R_\rho(z, -\Delta_N) = L^{cl}_{\rho,d} |\Omega| z^{\rho+d/2} + \frac{1}{4} L^{cl}_{\rho,d-1} |\partial\Omega| z^{\rho+(d-1)/2} + o(z^{\rho+(d-1)/2}), \qquad z \nearrow \infty. \tag{35}$$

These asymptotics in particular motivate the improvement of the Berezin–Li–Yau inequality by a negative remainder and the Kröger inequality by a positive remainder. Below we discuss such improvements and related open questions.

## 4.1 Dirichlet Laplacian

In 2003, Melas [24] obtained an improvement of the Berezin–Li–Yau inequality (10) for $\rho \geqslant 1$ by a negative geometric remainder. He showed that

$$R_\rho(z, -\Delta_D) \leqslant L^{cl}_{\rho,d} |\Omega| \left( z - C_d \frac{|\Omega|}{I(\Omega)} \right)_+^{\rho+d/2}, \quad z > 0, \quad \rho \geqslant 1, \tag{36}$$

where $C_d$ is a constant depending only on the dimension and $I(\Omega)$ denotes the *second moment of inertia* of $\Omega$:

$$I(\Omega) := \min_{a\in\mathbb{R}^d} \int_\Omega |x-a|^2 dx.$$

Notice that the power of $z$ in the second term is not of optimal order with respect to the asymptotic (34). This result was improved for $\rho \geqslant 1$ in dimension two by Kovařík, Vugalter, and Weidl [14]. They showed that the order of $z$ in the second term can be arbitrarily close to the optimal one. A first natural and interesting question is:

**Question 1.** *Can we get an improvement of the Berezin–Li–Yau inequality with a negative second term with the correct order of $z$?*

This question was answered by Weidl in 2008 [27] for $\rho \geqslant \frac{3}{2}$. He proved

$$R_\rho(z,-\Delta_D) \leqslant L^{cl}_{\rho,d}\,|\Omega|\,z^{\rho+d/2} - C_\Omega\, z^{\rho+(d-1)/2}, \qquad \rho \geqslant \frac{3}{2}. \tag{37}$$

Where $C_\Omega$ is a positive constant depending on geometry of $\Omega$. The geometric quantity $C_\Omega$ involves certain projections on $(d-1)$–dimensional hyperplanes. Later, Geisinger and Weidl [6] showed that $C_\Omega$ can be chosen depending only on the volume of $\Omega$. This result again holds for $\rho \geqslant 3/2$.
A second and more subtle question is:

**Question 2.** *can we get a uniform upper bound of the form*

$$R_\rho(z,-\Delta_D) \leqslant L^{cl}_{\rho,d}\,|\Omega|\,z^{\rho+d/2} - C(d,\rho)|\partial\Omega|\,z^{\rho+(d-1)/2}, \tag{38}$$

*where $C_d$ is a positive constant depending only on $d$ and $\rho$?*

The answer to this question is negative in general. Indeed, for any fixed $z>0$, we can increase $|\partial\Omega|$ without changing the individual eigenvalues $\lambda_k$ significantly so that the sum on the right-hand side becomes negative.

**Exercise 4.1.** *Construct an example which shows that inequality* (38) *is not true in general.*

Although having a uniform upper bound of the form (38) without any further condition on $\Omega$ is impossible, but we can modify the previous question by asking the following question.

**Question 3.** *Does inequality* (38) *hold for some families of domains in $\mathbb{R}^d$?*

This is a difficult question. Building on the previous developments, in 2011, Geisinger, Laptev and Weidl [5] investigated this question. The answer given in [5] was completed in a recent work of Larson [21]. In order to state their results, we first need to introduce some notations and definitions. Let $\mathbb{S}^{d-1} = \{x\in\mathbb{R}^d : |x|=1\}$ denote the unit-sphere in $\mathbb{R}^d$. For an arbitrary direction $v\in\mathbb{S}^{d-1}$ and $x\in\Omega$ we set

$$\begin{aligned} \theta(x,v) &= \inf\{t>0 : x+tv\notin\Omega\}\,,\\ d(x,v) &= \min\{\theta(x,v),\theta(x,-v)\}\,,\\ l(x,v) &= \theta(x,u)+\theta(x,-v)\,. \end{aligned}$$

**Theorem 6** ([5])**.** *Let $\Omega\subset\mathbb{R}^d$ be an open set and let $u\in\mathbb{S}^{d-1}$ and $\rho\geqslant 3/2$. Then for all $z>0$ the following inequality holds.*

$$R_\rho(z,-\Delta_D) \leqslant L^{cl}_{\rho,d}\int_\Omega \left(z-\frac{1}{4\,d(x,v)^2}\right)_+^{\rho+d/2} dx\,. \tag{39}$$

One can take infimum over all direction $v\in\mathbb{S}^{d-1}$:

$$l_0 = \inf_{v\in\mathbb{S}^{d-1}}\sup_{x\in\Omega} l(x,v)\,. \tag{40}$$

It is easy to check that $l_0 \geqslant 2d(x,v)$. Therefore, for $\rho \geqslant 3/2$ and all $z>0$ we get

$$R_\rho(z) \leqslant L^{cl}_{\rho,d}\,|\Omega| \left(z-\frac{1}{l_0^2}\right)_+^{\rho+d/2}. \tag{41}$$

For a convex domain, $l_0$ is equal to the width of the domain. The width of a domain $\Omega$ is defined as

$$\omega(\Omega) := \sup_{v\in\mathbb{R}^d} \omega_v(\Omega),$$

where

$$\omega_v(\Omega) := \sup\{v\cdot(x-y) : x,y\in\Omega\} = \max\{v\cdot(x-y) : x,y\in\partial\Omega\}. \tag{42}$$

The proof of Theorem 6 and also the previous results in [27] by Weidl are based on an inductive argument which allows to reduce the problem to estimating the Riesz mean $R_\rho(z)$ of the one-dimensional Dirichlet Laplacian on an open interval [27]. It is also need to use a sharp Lieb–Thirring inequalities for Schrödinger operators obtained in [19]. The proof of results in [5] follows similar steps as in [27]. Their main improvement was to get a more geometric and better bound for the Riesz mean of the Dirichlet Laplacian on a finite open interval $I$. More precisely they proved that on a finite open interval $I\subset\mathbb{R}$, the following inequality holds for the Riesz mean of the Dirichlet Laplacian $-\frac{d^2}{dt^2}$.

$$R_\rho(z,-\frac{d^2}{dt^2}) \leqslant L^{cl}_{\rho,1}\int_I \left(z-\frac{1}{4\delta(t)^2}\right)_+^{\rho+1/2} dt \qquad \rho\geqslant 1$$

where $\delta(t)$ denotes the distance to the boundary of $I$. This improvement leads to a nice description of the second term in the upper bound in terms of the geometry near the boundary.
Our goal is not to go to any details of the proof. We refere the interested reader to [5, 21] and the references therein for the details of the proof.

**Exercise 4.2.** *Having an improved version of the Berezin–Li–Yau inequality, one can get an improvement on bounds for other related spectral quantities via some integral transform.*

1. *Apply the Laplace transform to* (41) *and get the following improvement of of the Kac inequality*

$$\sum_{j=1}^{\infty} e^{-\lambda_j t} \leqslant \frac{|\Omega|}{(4\pi t)^{d/2}}\, e^{-\frac{t}{l_0^2}}. \tag{43}$$

2. *Apply the Mellin transform to* (43) *and get the following estimate for* $\zeta_{spec}(\rho)$.

$$\zeta_{spec}(\rho) \leqslant \frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(\rho-d/2)}{\Gamma(\rho)}\,|\Omega|\,\left(\frac{1}{l_0^2}\right)^{\frac{d}{2}-\rho}. \tag{44}$$

Now we describe how Geisinger, Laptev and Weidl related $d(x,v)$ to the geometry near the boundary. For $x\in\Omega$ let

$$\Omega(x) \;=\; \{y\in\Omega \,:\, x+t(y-x)\in\Omega\,,\ \forall\, t\in[0,1]\}\,,$$

and let

$$\delta(x) \;=\; \inf\left\{|y-x| \,:\, y\notin\overline{\Omega(x)}\right\}$$

denote the distance to the exterior of $\Omega(x)$.
Note that if you place a light source at $x$, then $\Omega(x)$ is the part of $\Omega$ that lights up.

For fixed $\varepsilon>0$ set

$$A_\varepsilon(x) \;=\; \left\{a\in\mathbb{R}^d\backslash\overline{\Omega(x)} \,:\, |x-a|<\delta(x)+\varepsilon\right\}$$

and for $a\in A_\varepsilon(x)$ set $B_x(a)=\{y\in\mathbb{R}^d \,:\, |y-a|<|x-a|\}$ and

$$\rho(x) \;=\; \inf_{\varepsilon>0}\sup_{a\in A_\varepsilon(x)} \frac{|B_x(a)\backslash\overline{\Omega(x)}|}{\omega_d|x-a|^d}\,,$$

where $\omega_d$ denotes the volume of the unit ball in $\mathbb{R}^d$.

Note that $\mathbb{R}^d\backslash\overline{\Omega(x)}$ is open, hence $\rho(x) > 0$ hold for $x \in \Omega$. Finally, we define

$$M_\Omega(z) \;=\; \int_{R_\Omega(z)} \rho(x)\, dx\,,$$

where $R_\Omega(z) \subset \Omega$ denotes the set $\{x \in \Omega \,:\, \delta(x) < 1/(4\sqrt{z})\}$. This is like a tube of width $1/(4\sqrt{z})\}$ around the boundary. If we define

$$\Omega_t := \{x \in \Omega : \mathrm{dist}(x, \Omega^c) \geqslant t\},$$

then $R_\Omega(z) = \Omega\backslash\Omega_{1/(4\sqrt{z})}$. We can now state a more geometric interpretation of the remainder term in (39):

**Theorem 7.** *Let $\Omega \subset \mathbb{R}^d$ be an open set with finite volume and $\rho \geqslant 3/2$. Then for all $z > 0$ we have*

$$R_\rho(z, -\Delta_D) \;\leqslant\; L^{cl}_{\rho,d}\,|\Omega| z^{\rho+d/2} - L^{cl}_{\rho,d}\, 2^{-d+1}\, z^{\rho+d/2}\, M_\Omega(z)\,. \tag{45}$$

The function $\rho(x)$ depends on the behaviour of the boundary close to $x \in \Omega$. For example, $\rho(x)$ is small close to a cusp. For a convex domain $\rho(x)$ is larger than $1/2$.

In particular, if the dimension of the boundary equals $d-1$, we have

$$M_\Omega(z) = |\partial\Omega|\, z^{-1/2} + o(z^{-1/2})$$

as $z \to \infty$, and in the limit the second term in (45) is the same as in the second term of the in (34) up to a constant depending only on $\rho$ and $d$.

To see how all these definitions and notations leads to get inequality (45) from inequality (39), we present the proof of Theorem 7 in [5].

*Proof of Theorem 7.* By averaging (39) we get

$$R_\rho(z, -\Delta_D) \;\leqslant L^{cl}_{\rho,d}\, z^{\rho+d/2} \int_\Omega \int_{\mathbb{S}^{d-1}} \left(1 - \frac{1}{4\, z\, d(x,v)^2}\right)_+^{\rho+d/2} d\nu(v)\, dx\,,$$

where $d\nu(v)$ denotes the normalised measure on $\mathbb{S}^{d-1}$.

For $x \in \Omega$ and $a \notin \overline{\Omega(x)}$ let us define

$$\Theta(x,a) := \left\{u \in \mathbb{S}^{d-1} : x + su \in B_x(a)\backslash\overline{\Omega(x)},\ \text{for some } s > 0\right\}.$$

For such $s$ and $u \in \Theta(x,a)$ we have

$$d(x,u) \leqslant s \;\leqslant\; 2\,|x-a|\,. \tag{46}$$

Let

$$\rho_a(x) \;=\; \frac{|B_x(a)\backslash\overline{\Omega(x)}|}{\omega_d |x-a|^d}\,,$$

By definition of $\rho_a(x)$ and $\Theta(x,a)$ we get

$$\rho_a(x)\,\omega_d |x-a|^d \;=\; |B_x(a)\backslash\overline{\Omega(x)}| \;\leqslant\; \int_{\Theta(x,a)} d\nu(u)\,\omega_d (2|x-a|)^d\,,$$

hence

$$\int_{\Theta(x,a)} d\nu(u) \;\geqslant\; 2^{-d}\,\rho_a(x)\,. \tag{47}$$

Having $d(x,u) = d(x,-u)$, we get the following estimate.

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \left(1 - \frac{1}{4zd(x,u)^2}\right)_+^{\rho+d/2} d\nu(u) \;&=\; 2\int_{\mathbb{S}^{d-1}_+} \left(1 - \frac{1}{4zd(x,u)^2}\right)_+^{\rho+d/2} d\nu(u)\\ &\leqslant\; 1 - 2\int_{\left\{u\in\mathbb{S}^{d-1}_+\,:\, d(x,u)\leqslant 1/(2\sqrt{z})\right\}} d\nu(u)\,.\end{aligned}$$

Now fix $z>0$ and choose $0<\varepsilon<1/(4\sqrt{z})$ and $a\in A_\varepsilon(x)$. By definition of $A_\varepsilon(x)$ it follows that for all $u\in\Theta(x,a)$

$$d(x,u)\ \leqslant\ 2|x-a|\ <\ 2(\delta(x)+\varepsilon)\,. \tag{48}$$

Assume that $\delta(x)\leqslant 1/(4\sqrt{z})-\varepsilon$. Notice that $\Theta(x,a)$ must be contained in one hemisphere of $\mathbb{S}^{d-1}$ which we denote by $\mathbb{S}^{d-1}_+$. From (48) it follows that

$$\Theta(x,a)\subset\left\{u\in\mathbb{S}^{d-1}_+\ :\ d(x,u)\leqslant 1/(2\sqrt{z})\right\}\,.$$

Using (47), we conclude

$$\int_{\mathbb{S}^{d-1}}\left(1-\frac{1}{4zd(x,u)^2}\right)_+^{\rho+d/2}d\nu(u)\ \leqslant\ 1-2\int_{\Theta(x,a)}d\nu(u)\ \leqslant\ 1-2^{1-d}\rho_a(x)\,.$$

Since $a\in A_\varepsilon(x)$ was arbitrary we arrive at

$$\int_{\mathbb{S}^{d-1}}\left(1-\frac{1}{4zd(x,u)^2}\right)_+^{\rho+d/2}d\nu(u)\ \leqslant\ 1-2^{1-d}\rho(x)\,,$$

for all $x\in\Omega$ with $\delta(x)\leqslant 1/(4\sqrt{z})-\varepsilon$ and we can take the limit $\varepsilon\to 0$.

We conclude that

$$\begin{aligned}\int_\Omega\int_{\mathbb{S}^{d-1}}\left(1-\frac{1}{4zd(x,u)^2}\right)_+^{\rho+d/2}d\nu(u)\,dx\ &\leqslant\ |\Omega|-2^{1-d}\int_{\{x\in\Omega\,:\,\delta(x)<1/(4\sqrt{z})\}}\rho(x)\,dx\\ &=\ |\Omega|-2^{1-d}M_\Omega(z).\end{aligned}$$

□

Note that if $\Omega$ is a convex domain then as mentioned above $\rho(x)>1/2$. Therefore,

$$M_\Omega(z)\ =\ \int_{R_\Omega(\Lambda)}\rho(x)\,dx\ >\ \int_{R_\Omega(z)}\frac{1}{2}\,dx\ =\ \frac{1}{2}\int_0^{1/(4\sqrt{z})}|\partial\Omega_t|\,dt. \tag{49}$$

As an immediate corollary (see [5, Corollary 3.4]), one can get

$$R_\rho(z,-\Delta_D)\ \leqslant\ L^{cl}_{\rho,d}\,|\Omega|z^{\rho+d/2}-L^{cl}_{\rho,d}\,2^{-d-2}\,z^{\rho+(d-1)/2}\,|\partial\Omega_{1/(4\sqrt{z})}|\,. \tag{50}$$

Geisinger, Laptev and Weidl in [5] conjecture that for any convex domain $\Omega\subset\mathbb{R}^2$ the following inequality holds

$$|\partial\Omega_t|\ \geqslant\ \left(1-\frac{3t}{\omega(\Omega)}\right)_+|\partial\Omega|\,,$$

where $\omega(\Omega)$ is the width of $\Omega$. The above inequality is true for a large family of convex domains including regular polygons. Under this assumption they proved that for $\rho\geqslant 3/2$ we have $R_\rho(z)=0$ if $z\leqslant\pi^2/\omega(\Omega)^2$ and

$$R_\rho(z)\ \leqslant\ L^{cl}_{\rho,2}\,|\Omega|\,z^{\rho+1}-C(\rho)\,|\partial\Omega|\,z^{\rho+1/2}\quad\text{if}\quad z>\pi^2/\omega(\Omega)^2\,.$$

Later, Larson [20] proved the conjecture for any convex domain in $\mathbb{R}^d$:

**Theorem 4.1.** *[20] Let $\Omega\subset\mathbb{R}^d$ be a convex domain with inradius $r$. Then, for any inner parallel set $\Omega_t$, $t\geqslant 0$, the following inequality holds*

$$|\partial\Omega_t|\geqslant\left(1-\frac{t}{r}\right)_+^{d-1}|\partial\Omega|. \tag{51}$$

By Theorem 4.3, we can estimate $M_\Omega(z)$ as follows.

$$M_\Omega(z)\ >\ \frac{|\partial\Omega|}{2}\int_0^{1/(4\sqrt{z})}\Big(1-\frac{t}{r}\Big)_+^{d-1}dt\ =\ \frac{|\partial\Omega|}{8\sqrt{z}}\int_0^1\Big(1-\frac{s}{4r\sqrt{z}}\Big)_+^{d-1}ds. \tag{52}$$

This leads to the following theorem,

**Theorem 8.** *[21] Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain with inradius $r$. Then for every $\rho \geqslant 3/2$, there exists a constant $C(\rho, n)$ such that for $z \geqslant \frac{\pi^2}{4r^2}$*

$$R_\rho(z, -\Delta_D) \leqslant L^{cl}_{\rho,d}|\Omega| z^{\rho+d/2} - C(\rho, d) L^{cl}_{\rho,(d-1)}|\partial\Omega| z^{\rho+(d-1)/2}, \tag{53}$$

*and if $z \leqslant \frac{\pi^2}{4r^2}$ then*

$$R_\rho(z, -\Delta_D) = 0.$$

The results above hold for $\rho \geqslant 3/2$. So the following question remains open.

**Open Question 1.** *Can such improved estimates be extended to $1 \leqslant \rho < 3/2$?*

**Exercise 4.3.** *Let $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subset \mathbb{R}^{d_1}$ is a bounded, convex domain with inradius $r$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ is bounded and open. Assume that $\rho + d_2/2 \geqslant 3/2$ and that for all $z > 0$*

$$R_\rho(z) \leqslant L^{cl}_{\rho,d_2}|\Omega_2| z^{\rho+d_2/2}. \tag{54}$$

*Then for $z > \frac{\pi^2}{4r^2}$ we have that*

$$R_\rho(z) \leqslant L^{cl}_{\rho,d}|\Omega| z^{\rho+n/2} - C(n_1, \rho + d_2/2) L^{cl}_{\rho,n-1}|\Omega_2|\,|\partial\Omega_1| z^{\rho+(n-1)/2}, \tag{55}$$

*where $d = d_1 + d_2$ and $C(\rho, d)$ is the same constant as in Theorem 8,*
*and $R_\rho(z) = 0$ if $z > \frac{\pi^2}{4r^2}$.*

Notice that if $d_2 \geqslant 3$ and $\Omega_2$ is tiling in the above exercise, then we obtain an improved version of the Pólya conjecture for $\Omega$.

## 4.2 Neumann Laplacian

The asymptotic (35) suggests a possibility of improving the Kröger inequality by adding a second positive term to the lower bound of $R_\rho(z, -\Delta_N)$. Recently, Harrell and Stubbe [10] obtained such improvement of the Kröger estimate (15). The order of $z$ in the second remainder term is the same as in the two-term asymptotic (35). More precisely they proved:

**Theorem 9.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d \geqslant 2$, with Lipschitz boundary. Then for every $z \geqslant 0$*

$$R(z, -\Delta_N) \geqslant L^{cl}_{1,d}|\Omega| z^{\frac{d}{2}+1} + \frac{1}{4} L^{cl}_{1,d-1} \frac{|\Omega|}{\omega_v(\Omega)} z^{\frac{d}{2}+\frac{1}{2}} - \frac{1}{96} (2\pi)^{2-d} \omega_d \frac{|\Omega|}{\omega_v(\Omega)^2} z^{\frac{d}{2}},$$

*where $\omega_v(\Omega)$ is the width of $\Omega$ in the direction of $v \in \mathbb{S}^{d-1}$ (see (42) for the definition of $\omega_v(\Omega)$).*

**Exercise 4.4.** *Using the Riesz iteration show that for any $\rho \geqslant 1$*

$$R_\rho(z, -\Delta_N) \geqslant L^{cl}_{\rho,d}|\Omega| z^{\frac{d}{2}+\rho} + L^{cl}_{\rho,d-1} \frac{|\Omega|}{4\omega_v(\Omega)} z^{\frac{d}{2}+\rho-\frac{1}{2}} - \frac{\pi}{96} L^{cl}_{\rho,d-2} \frac{|\Omega|}{\omega_v(\Omega)^2} z^{\frac{d}{2}+\rho-1}. \tag{56}$$

We note that $\omega_v(\Omega)$ always lies between twice the inradius and the diameter of $\Omega$.We refer to [10] on discussion on estimates for $\omega_v(\Omega)$.

**Remark 4.1.** *One can obtain an improved version of the Pólya conjecture for a product domain $\Omega = \Omega_1 \times \Omega_2$ when $\Omega_1$ satisfies the Pólya conjecture. For more details see [10, Corollary 1.4]*

Similar to the Dirichlet case the following natural question arises.

**Open Question 2.** *Can we prove an improved version of the Kröger inequality on a family of domains with a positive second term depending only $|\partial\Omega|$, $\rho$ and $d$?*

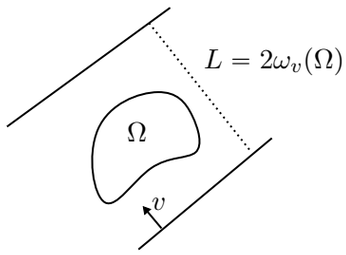
**Question 4.** *Can you give an example showing that the answer to the previous question is negative if we consider the family of all bounded Lipschitz domains?*

### Few words about the proof

The proof based on using the averaged variational principle togather with an averaging procedure. Here, we briefly explain the sketch of the proof. Recall the averaged variational principle

$$\sum_j (z-\mu_j)_+ \int_{\mathfrak{M}} |\langle \varphi_j, f_\xi\rangle|^2 d\mu \geqslant \int_{\mathfrak{M}_0} \left( z\int_\Omega |f_\xi(x)|^2 dx - \int_\Omega |\nabla f_\xi(x)|^2 dx\right) d\mu,$$

where $\{\phi_n\}$ is an orthonormal basis of eigenfunctions, see Theorem 4 for the statement.
Harrell and Stubbe proved that by choosing an appropriate family of test functions $f_\xi(x)$, one can get a two–term lower bound for the Riesz mean $R_\rho(z,-\Delta_N)$.
Let us fix a direction $v\in\mathbb{S}^{d-1}$ and consider a strip of width $L=2\omega_v(\Omega)$ perpendicular to the direction $v$ containing $\Omega$. After rotation we can assume that $v=e_d=(0,\cdots,0,1)$. Let $x=(x',x_d)$, $\xi=(\xi',\xi_d)\in\mathbb{R}^d$, where $x',\xi'\in\mathbb{R}^{d-1}$.




We consider the following family of test functions

$$f_{\xi',n}(x) = (2\pi)^{-\frac{d-1}{2}} e^{i\xi'\cdot x'}\psi_n(x_d),$$

where $\psi_n$ is an orthonormal basis of Neumann eigenfunctions on an interval of length $L$, which are

$$\psi_0(y)=L^{-1/2}, \qquad \psi_n(y)=\sqrt{\frac{2}{L}}\cos\left(\frac{\pi n y}{L}\right),$$

and their corresponding eigenvalues are $\frac{(\pi n)^2}{L^2}$, $n\in\mathbb{N}$. Now apply the the averaged variational formula to $f_{\xi',n}$. We get

$$\sum_j (z-\mu_j)_+ |\langle f_{\xi',n},\phi_j\rangle|^2 \geqslant (2\pi)^{1-d}(z-|\xi'|^2)\int_\Omega \psi_n(x_d)^2 - (2\pi)^{1-d}\int_\Omega \psi_n'(x_d)^2$$

By using the simple trigonometric identities $\cos^2 t = \dfrac{1+\cos 2t}{2}$ and $\sin^2 t = \dfrac{1-\cos 2t}{2}$ we obtain

$$\begin{aligned}\sum_j (z-\mu_j)_+ |\langle f_{\xi',n},\phi_j\rangle|^2 \quad &\geqslant \quad (2\pi)^{1-d}L^{-1}|\Omega|\left(z-|\xi'|^2-\frac{(\pi n)^2}{L^2}\right)\\ &\quad +(2\pi)^{1-d}L^{-1}\left(z-|\xi'|^2+\frac{(\pi n)^2}{L^2}\right)(1-\delta_{0,n})\int_\Omega \cos\left(\frac{2\pi n x_d}{L}\right),\end{aligned}$$

where $\delta_{0,n}$ denotes the Kronecker delta. We now set $\mathfrak{M}=\mathbb{R}^{d-1}\times\mathbb{N}$ and

$$\mathfrak{M}_0=\left\{(\xi',n)\in\mathbb{R}^{d-1}\times\mathbb{N} : |\xi'|^2+\frac{\pi^2 n^2}{L^2}\leqslant z\right\}.$$

After calculating the integrals and using the Parseval identity on the left-hand side (see the details in [10]) we get

$$\begin{aligned}\sum_j (z-\mu_j)_+ \quad &\geqslant \quad \frac{2}{d+2}(2\pi)^{-d}\omega_d|\Omega|\, z^{\frac{d}{2}+1}\\ &\quad +\frac{1}{d+1}(2\pi)^{1-d}\omega_{d-1}|\Omega|L^{-1}\, z^{\frac{d+1}{2}}\\ &\quad -\frac{1}{24}(2\pi)^{2-d}\omega_d|\Omega|L^{-2}z^{\frac{d}{2}}\\ &\quad +\int_{\mathfrak{M}_0}\sum (2\pi)^{1-d}\left(z-|\xi'|^2+\frac{(\pi n)^2}{L^2}\right)(1-\delta_{0,n})\int_\Omega \cos\left(\frac{2\pi n x_d}{L}\right).\end{aligned} \tag{57}$$

We need to estimate the last term. Take a strip so that $\Omega \subset \{x \in \mathbb{R}^d : 0 \leqslant x_d \leqslant L/2\}$. Then we calculate the estimate above for the translated domain $\Omega + \frac{L}{2} = \{(x', x_d + \frac{L}{2}) : x \in \Omega\}$. All terms in the above calculation remains unchanged unless the last term contains the following quantity

$$\cos\left(\frac{2\pi n x_d}{L}\right) \mapsto \cos\left(\frac{2\pi n (x_d + \frac{L}{2})}{L}\right).$$

We average our estimates on $\Omega + \frac{L}{2}$ and $\Omega$ and note that

$$\frac{1}{2}\left(\cos\left(\frac{2\pi n x_d}{L}\right) + \cos\left(\frac{2\pi n x_d + L/2)}{L}\right)\right) = \begin{cases} \cos\left(\frac{4\pi k x_d}{L}\right) & \text{if } n = 2k\ , \\ 0 & \text{if } n = 2k+1, \end{cases}$$

which is nonzero for $n$ even. If this time we translate $\Omega$ by $L/4$ and then do the average between estimate (57) on $\Omega + \frac{L}{4}$ and the previous averaged estimate, the last term contains only integers which are multiple of 4. If we continue this averaging procedure, then at some step $k$ the set $\mathfrak{M}_0$ does not contain any integer of multiple $2^k$. So we can drop the last term all together. This completes the proof.

The ideas and techniques discussed in this note, such as averaged variational principle [4, 10] and results in [16], can have further applications in the study of a large class of operators. Very recently, Laptev and myself [9] used some of these ideas to study bounds on the Riesz means of an operator with spectral parameter on the boundary (so-called the mixed Steklov problem). This in part motivated me to give a mini-course on this subject.

*"Mathematics only exists in a living community of mathematicians that spreads understanding and breaths life into ideas both old and new. The real satisfaction from mathematics is in learning from others and sharing with others. All of us have clear understanding of a few things and murky concepts of many more. There is no way to run out of ideas in need of clarification. The question of who is the first person to ever set foot on some square meter of land is really secondary. Revolutionary change does matter, but revolutions are few, and they are not self-sustaining — they depend very heavily on the community of mathematicians."*

— Bill Thurston

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